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Let R be a ring, and let M_R denote that M is a right R-module. In this paper, the unique maximal rational extension of R_R, denoted by $\bar{R}$, is shown to be a ring whose operations preserve the module structure in $(\bar{R})_R$. It is then shown that if the right singular ideal is zero, the injective envelope of R_R is the maximal rational extension and therefore is a ring. It is further shown that this ring is right self-injective and is a von Neumann ring.

These results are applied to the $n \times n$ upper triangular matrix rings to compute their injective envelopes.

ON RATIONAL EXTENSIONS OF A RING

by

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INTRODUCTION AND PRELIMINARY REMARKS

The purpose of this paper is to examine the rational extensions of a ring and to prove that under certain conditions the injective hull of a ring is a right self-injective, von Neumann ring. Furthermore, we shall show that the module structure of the injective hull induces the ring multiplication.

In the first section of Chapter I, we define a rational extension of a module over a ring. This concept is closely related to essential extensions of modules in that every rational extension of a module is an essential extension. Several theorems that characterize rational extensions are proved. Using these theorems, we show that when M_R is a right R-module with zero right singular submodule, every essential extension of M_R is a rational extension.

In section 2, the concept of maximal rational extension is introduced. We prove that any right R-module M_R has a maximal rational extension unique up to isomorphism, and we denote it by $\overline{M}$. For any ring R with unity, the maximal rational extension $\overline{R}$ of R is shown to be a ring whose operations preserve the module structure in $\overline{R}$. We show, using theorems established in section 1, that if R is a ring with zero right singular ideal, then the injective hull of R_R is $(\overline{R})_R$ and therefore is a ring.

We further characterize this ring in section 3. After defining right quotient rings and von Neumann rings, we show for any ring R with zero right singular ideal that the injective hull E of R_R is the maximal right quotient ring of R. We then prove $\Lambda = \mathrm{Hom}_R(E,E)$ is a right self-injective, von Neumann ring. We conclude the chapter by showing that when the right singular ideal of a ring is zero, the injective hull of R is ring isomorphic to Λ, and is therefore a right self-injective, von Neumann ring.

The results of Chapter I are used in Chapter II to show that the injective hull of the $n \times n$ upper triangular matrix ring over a division ring D is the full ring of $n \times n$ matrices over D.

Throughout this paper, the term "ring" will mean "ring with unity".

We define a <u>right R-module</u> as an abelian group $(M,+)$ together with a function $\mu : M \times R \to M$, where we let $\mu(a,x) = a\mu x$ such that:

(a) $x\mu(a+b) = x\mu a + x\mu b$ for all $a,b \in R$ and $x \in M$;

(b) $(x+y)\mu a = x\mu a + y\mu a$ for all $x,y \in M$ and $a \in R$;

(c) $x\mu(ab) = (x\mu a)\,\mu\, b$ for all $a,b \in R$ and $x \in M$;

(d) $x\mu 1 = x$ for all $x \in M$.

We shall denote the right R-module M by M_R. If N is a subgroup of M, then N is itself a candidate for a right R-module. We shall call a subgroup N of M a <u>submodule of M_R</u> whenever $x\mu a$ belongs to N for all $a \in R$ and $x \in N$. This will be denoted by $N_R \subseteq M_R$ or $M_R \supseteq N_R$.

If M_R and L_R are right R-modules, a function $f : M \to L$ is called a right R-homomorphism from M into L if in addition to being a group homomorphism it satisfies the following: $f(x\mu_1 a) = f(x)\mu_2 a$ for all $a \in R$ and $x \in M$, where μ_1 and μ_2 represent the module operations for M and L respectively. We shall denote the set of all right homomorphisms from a right R-module A_R into a right R-module B_R by $\mathrm{Hom}_R(A,B)$.

A submodule N_R of M_R is said to be an essential submodule of M_R provided if for every submodule $0 \neq L_R$ of M_R, $N \cap L \neq 0$. In case N_R is an essential submodule of M_R, we say M_R is an essential extension of N_R and denote it by $M \nabla N$.

A module M_R is said to be R-injective provided for every exact sequence of R-modules $0 \to A \xrightarrow{f} B$, that is $f : A \to B$ is a one-to-one R-homomorphism, and for every $\phi \in \mathrm{Hom}_R(A,M)$, there exists a $\psi \in \mathrm{Hom}_R(B,M)$ such that $\phi = \psi \circ f$.

A result known as Baer's Lemma is used in this paper to check the injectivity of a module and is stated as follows:

Baer's Lemma. For a module Q_R, the following are equivalent:

(1) Q_R is injective;

(2) For every right ideal I of R and for every R-homomorphism $\phi : I \to Q$, there exists a $\phi' \in \mathrm{Hom}_R(R,Q)$ such that $\phi'(a) = \phi(a)$ for all $a \in I$;

(3) For every right ideal I of R and for every R-homomorphism $\phi : I \to Q$, there is a $q \in Q$ such that $\phi(a) = qa$ for all $a \in I$.

We shall call an injective essential extension of a module M_R the <u>injective envelope</u> or <u>injective hull</u> of M_R and denote it by $E(M)$. It is a well-known result that every module has an injective essential extension, and if E and E' are two injective essential extensions of M, then there exists an R-isomorphism $f : E \to E'$ such that $f(x) = x$ for every $x \in M$. Thus the injective envelope of a module is unique up to isomorphism.

CHAPTER I

THE INJECTIVE ENVELOPE CONSIDERED AS A RATIONAL EXTENSION

Section 1: Rational and Essential Extensions

1.1.1 DEFINITION. If $M_R \supseteq N_R$, then M_R is a rational extension of N_R, denoted by $M \blacktriangledown N$, in case for each submodule B_R of M_R such that $N_R \subseteq B_R$, $f \in \mathrm{Hom}_R(B,M)$ satisfies $f(N) = 0$ if and only if $f = 0$.

1.1.2 LEMMA. If $M \blacktriangledown N$, then $M \nabla N$.

Proof: Let M_R and N_R be right R-modules such that $M \blacktriangledown N$. Let F_R be a submodule of M_R. Suppose $F \cap N = 0$. It suffices to show $F_R = 0$. Since $F \cap N = 0$, $B = F + N$ is a direct sum. Let $a \in B$. Then there exist $x \in F_R$ and $y \in N_R$ such that $a = x + y$. So if $\pi_F(a) = \pi_F(x+y) = x$, then $\pi_F \in \mathrm{Hom}_R(B,M)$ and it is clear that $\pi_F(N) = 0$. Therefore, since $M \blacktriangledown N$, $\pi_F = 0$. This implies $F = 0$ since $F = \mathrm{Im}\ \pi_F$.

1.1.3 LEMMA. $M \blacktriangledown N$ if and only if for each $x \in M$ and $0 \neq y \in M$, there exist $r \in R$ and $n \in Z$ such that $xr + xn \in N$ and $yr + yn \neq 0$.

Proof: $(\leftarrow)$ Let B_R be a submodule of M_R such that $M_R \supseteq B_R \supseteq N_R$ and f be an element of $\mathrm{Hom}_R(B,M)$ such that $f(N) = 0$. We need to show $f = 0$; so assume $f \neq 0$. Then, there exists an $x \in B$ such that $y = f(x) \neq 0$. By hypothesis, there is an $r \in R$ and $n \in Z$ such that $xr + xn \in N$ and

$yr + yn \neq 0$. However, $0 = f(xr+xn) = f(xr) + f(xn) = f(x) \cdot r + f(x) \cdot n = yr + yn$. This is a contradiction. Therefore, $f = 0$.

($\leftarrow$) Assume $M \blacktriangledown N$. Let x and $0 \neq y$ be elements of M_R. Let $x_N = \{(r,n) \mid (r,n) \in R \times Z$ and $xr + nx \in N\}$, and $y_0 = \{(r,n) \mid (r,n) \in R \times Z$ and $yr + ny = 0\}$. It is sufficient to show that $x_N \not\subseteq y_0$; that is, there exists $(r,n) \in x_N$ with $(r,n) \notin y_0$. To this end, we shall show if $y_0 \supseteq x_N$, then $y = 0$, and thus arrive at a contradiction since $y \neq 0$.

Assume $y_0 \supseteq x_N$. Let B denote the submodule of M which consists of elements of the form $a + x(r,n)$ for some $a \in N$ and $(r,n) \in R \times Z$ where $x(r,n) = xr + xn$; thus, $B = N + xR + xZ$. Consider the relation f between B and M defined by $f(a+x(r,n)) = y(r,n)$. To show f is a function, assume $a + x(r,n) = a' + x(r',n')$. Then $x(r-r',n-n') = a - a' \in N$. So, $(r-r',n-n') \in x_N$. Since $y_0 \supseteq x_N$, $yr - yr' + yn - yn' = 0$. This implies $yr + yn = yr' + yn'$. But $f(a+x(r,n)) = yr + yn$ and $f(a'+x(r',n')) = yr' + yn'$. Hence, $f(a+x(r,n)) = f(a'+x(r',n'))$. Now we need to show f is an R-homomorphism. Let b_1 and b_2 be elements of B. There exist $a_1, a_2 \in N$, $r_1, r_2 \in R$, and $n_1, n_2 \in Z$ such that $b_1 = a_1 + x(r_1, n_1)$ and $b_2 = a_2 + x(r_2,n_2)$. One property of an R-homomorphism is satisfied since $f(b_1+b_2) = y(r_1+r_2,n_1+n_2) = y(r_1,n_1) + y(r_2,n_2) = f(b_1) + f(b_2)$. Now if $r \in R$, $f(b_1r) = f([a_1+x(r_1,n_1)]\cdot r) = f(a_1r+xr_1r+n_1xr) = f(a_1r) + f(xr_1r) + f(n_1xr) =$

$f(a_1r+x(0,0)) + f(0+x(r_1r,0)) + n_1f(0+x(r_1,0)) = 0 + y(r_1r,0) + n_1y(r,0) = yr_1r + n_1yr = (yr_1+n_1y)r = y(r_1,n_1)r = f(b)r$. Therefore, $f \in Hom_R(B,M)$.

Let $a \in N$. Then $f(a) = f(a+x(0,0)) = 0$ and so $f(N) = 0$. Therefore, $f = 0$. This implies $y(r,n) = 0$ for all $(r,n) \in R \times Z$ and this means $y = 0$. So we know if $y_0 \supseteq x_N$, then $y = 0$. Since $y \neq 0$, $y_0 \not\supseteq x_N$ and there exists $(r,n) \in R \times Z$ such that $xr + xn \in N$ and $yr + ny \neq 0$. This proves the theorem.

<u>1.1.4 LEMMA</u>. If $M \blacktriangledown N$, then for each submodule F_R of M_R, $F \blacktriangledown (F \cap N)$.

Proof: Let $M \blacktriangledown N$ and $F_R \subseteq M_R$. Let $x \in F$ and $0 \neq y \in F$. Then there is an $r \in R$ and $n \in Z$ such that $xr + nx \in N$ and $yr + ny \neq 0$. But $xr + nx$ is also in F because $x \in F$. Therefore, $xr + nx \in F \cap N$ and $yr + ny \neq 0$. By 1.1.3, $F \blacktriangledown (F \cap N)$.

<u>1.1.5 DEFINITION</u>. For a module M_R, let $Z(M) = \{x \in M \mid (0:x)$ is an essential right ideal of $R\}$. We call $Z(M)$ the <u>singular submodule of M_R</u>.

The following lemma shows $Z(M)$ is a submodule of M_R, and in the case when $M_R = R_R$, $Z(M)$ is a two-sided ideal of R and is called the <u>right singular ideal of R</u>.

<u>1.1.6 LEMMA</u>. $Z(M)$ is a submodule of M_R and $Z(R)$ is a two-sided ideal of R.

Proof: Let M_R be a right R-module and let $x,y \in Z(M)$ and $r \in R$. We need to show $x-y \in Z(M)$ and $xr \in Z(M)$. First, we show $x-y \in Z(M)$. Choose $k \in (0:x) \cap (0:y)$. Then $xk = 0 = yk$. Hence, $(x-y)k = 0$ and $k \in (0:x-y)$. Therefore, $(0:x-y) \supseteq (0:x) \cap (0:y)$. Let L be a non-zero right ideal of R. Then $((0:x) \cap (0:y)) \cap L = (0:x) \cap ((0:y) \cap L)$ which is not equal to zero since $(0:y) \cap L \neq 0$ and $R \nabla (0:x)$. Therefore, $(0:x) \cap (0:y)$ is essential in R. Since $(0:x-y) \supseteq (0:x) \cap (0:y)$, $(0:x-y)$ is essential in R and $(x-y) \in Z(M)$. To finish the first part of the proof, let $k \in [(0:x):r]$. Then $rk \in (0:x)$ and $x(rk) = (xr)k = 0$ which implies $k \in (0:xr)$. Hence, $(0:xr) \supseteq ((0:x):r)$. Now we shall show that $((0:x):r)$ is essential in R. Let K be a nonzero right ideal of R. Show $K \cap [(0:x):r] \neq 0$.

(Case 1): $r \cdot K = 0$. If $r \cdot k = 0$ for all $k \in K$, then $x(rk) = 0$ for all $k \in K$ which implies $0 \neq K \subset [(0:x):r]$.

(Case 2): $r \cdot K \neq 0$. Since rK is a right ideal of R, $(0:x) \cap rK \neq 0$. Therefore, there is a $0 \neq rk \in rK \cap (0:x)$. But $rk \in (0:x)$ implies $0 = x(rk)$ and that $k \in ((0:x):r)$. Clearly $0 \neq k \in [(0:x):r] \cap K$.

In either case, $((0:x):r)$ meets K nontrivially. Hence $((0:x):r)$ is essential in R. Since $(0:xr) \supseteq ((0:x):r)$, $(0:xr)$ is also essential in R. Therefore, $xr \in Z(M)$ and $Z(M)$ is a submodule of M_R.

By the preceding, $Z(R)$ is a right ideal of R. But if $x \in Z(R)$ and $a \in R$, $(0:ax) \supseteq (0:x)$ which implies $ax \in Z(R)$. Therefore, $Z(R)$ is a two-sided ideal of R.

<u>1.1.7 LEMMA</u>. Let $M_R \supseteq N_R$.

(1) $Z(N) = Z(M) \cap N$, and if $M \nabla N$, $Z(M) \nabla Z(N)$;

(2) if $M \nabla N$, then $Z(M) = 0$ if and only if $Z(N) = 0$.

Proof: (1) Let $x \in Z(N)$. Then $x \in N$ implies $x \in M$ which implies $x \in Z(M)$ since $R \nabla (0:x)$. Therefore, $x \in Z(M) \cap N$ and $Z(N) \subset Z(M) \cap N$. Let $x \in Z(M) \cap N$. Then $x \in N$ and $R \nabla (0:x)$ which implies $x \in Z(N)$. Therefore, $Z(M) \cap N \subset Z(N)$ and hence, $Z(N) = Z(M) \cap N$. Assume $M \nabla N$; we shall show $Z(M) \nabla Z(N)$. Let $0 \neq S_R$ be a submodule of M_R and $0 \neq K_R$ be a submodule of S_R. Then $K \cap N \neq 0$ and $0 \neq K \cap N = (K \cap S) \cap N = K \cap (S \cap N)$. Hence, $S \nabla (S \cap N)$ for each submodule S_R of M_R. So, $Z(M) \nabla (Z(M) \cap N)$ and since $Z(M) \cap N = Z(N)$, $Z(M) \nabla Z(N)$. Part (2) is immediate.

<u>1.1.8 THEOREM</u>. If $M_R \supseteq N_R$ and $Z(N) = 0$, then $M \nabla N$ if and only if $M \blacktriangledown N$.

Proof: ($\leftarrow$) 1.1.2

($\rightarrow$) Assume $M \nabla N$ and $Z(N) = 0$. We will show $M \blacktriangledown N$ by using 1.1.3. By 1.1.7, $Z(N) = 0$ implies $Z(M) = 0$. Let x and $0 \neq y \in M_R$. Let $(N:x) = \{r \in R \mid xr \in N\}$ and $(0:y) = \{r \in R \mid yr = 0\}$. We know $(0:y)$ is not an essential ideal of R since $Z(M) = 0$ and $y \neq 0$. If we can show $(N:x)$ is essential in R, we will know $(0:y) \not\supseteq (N:x)$, and hence there will exist an $r \in (N{:}x)$ such that $r \notin (0{:}y)$; that is, $xr \in N$ and $yr \neq 0$. Thus, the theorem will be proved. First, we need to show $(N{:}x)$ is a right ideal of R. Let $r_1, r_2 \in (N{:}x)$. Since M is a R-module, $x(r_1 - r_2) = xr_1 - xr_2$. But because N is a R-module, xr_1 and $xr_2 \in N$. Therefore, $xr_1 - xr_2 \in N$ and $r_1 - r_2 \in (N{:}x)$. The product $x(r_1 \cdot r_2) = (xr_1)r_2 \in N$ since $r_1 \in (N{:}x)$ implies $xr_1 \in N$. Therefore, $(N{:}x)$ is a subring of R. To show $(N{:}x)R \subseteq (N{:}x)$, let $r_1 \in (N{:}x)$ and $r_2 \in R$. By the previous argument, $x(r_1 \cdot r_2) = (xr_1)r_2 \in N$ and $r_1 \cdot r_2 \in (N{:}x)$. We now show $(N{:}x)$ is essential in R. Let $0 \neq s \in R$; then xsR is a submodule of M_R. If $xs = 0$, $s \in (N{:}x)$ and $0 \neq s \in sR \cap (N{:}x)$. If $xs \neq 0$, then $xsR \neq 0$ and $N \cap xsR \neq 0$. So there is an element a in R such that $0 \neq xsa \in N$. Therefore, $0 \neq sa \in (N{:}x)$ and since $sa \in sR$, $sR \cap (N{:}x) \neq 0$. In either case, $(N{:}x) \cap sR \neq 0$, and so $R \nabla (N{:}x)$.

When $Z(R) = 0$, this relationship between rational and essential extensions is very useful. In particular, when $Z(R) = 0$, we know the injective envelope of R_R is also a rational extension.

<u>1.1.9 LEMMA</u>. If $F \blacktriangledown M$, and F_R is R-isomorphic to K_R under a R-isomorphism ϕ, then $K \blacktriangledown \phi(M)$.

Proof: Suppose $F \blacktriangledown M$ and ϕ is an onto, one-to-one R-homomorphism of F into K. Let B_R be a submodule of K_R such that $K_R \supseteq B_R \supseteq \phi(M)$. Suppose $f \in \mathrm{Hom}_R(B,K)$ such that $f(\phi(M)) = 0$; we must show $f(B) = 0$. Since $B \supseteq \phi(M)$ and ϕ

is one-to-one, $\phi^{-1}(B) \supseteq M$. Let k be the restriction of $\phi^{-1}f\phi$ to $\phi^{-1}(B)$. Then $k \in \mathrm{Hom}_R(\phi^{-1}(B),F)$ and $k(M) = 0$ since $k(M) = \phi^{-1}(f(\phi(M))) = \phi^{-1}(0) = 0$. However, $F \blacktriangledown M$ implies that $k = 0$ which implies $k(\phi^{-1}(B)) = 0$. But this means $(\phi^{-1}f\phi\phi^{-1})(B) = \phi^{-1}(f(B)) = 0$, and since ϕ^{-1} is 1-1, that $f(B) = 0$. Therefore, $K \blacktriangledown \phi(M)$.

Section 2: When is the Injective Envelope a Ring?

1.2.1 DEFINITION. If $G_R \supseteq M_R$, then G_R is a maximal rational extension of M_R provided:

(a) G_R is a rational extension of M_R;

(b) if $F_R \supseteq M_R$ and $F \blacktriangledown M$, then the identity map from M into M can be extended to a monomorphism of F into G;

(c) if $F_R \supseteq G_R$ and $F \blacktriangledown M$, then $F = G$.

One might ask if every right R-module has a maximal rational extension. The answer is yes, and the following theorem shows the existence of such a maximal rational extension which is unique up to isomorphism.

1.2.2 THEOREM. Let E be the injective hull of M_R for some right R-module M_R. Let $\Lambda = \mathrm{Hom}_R(E,E)$ and $M^{\Lambda} = \{\lambda \in \Lambda \mid \lambda(M) = 0\}$. Then $\overline{M} = \cap\{\ker \lambda \mid \lambda \in M^{\Lambda}\}$ is a maximal rational extension of M_R, and $\overline{M}$ contains each rational extension of M_R that is contained in E. Also, if G_R is any maximal rational extension of M_R, then the canonical injection of M_R into $\overline{M}$ can be extended to a monomorphism of G onto $\overline{M}$.

Proof: The proof is divided into five parts which are collectively a proof of the theorem. Let M and $\overline{M}$ be defined as stated in the theorem.

(1) Show $\overline{M} \blacktriangledown M$. We know $\overline{M}$ is an R-module since it is the intersection of the kernels of R-homomorphisms. It is clear that

$M_R \subseteq \overline{M}_R$. Let $B_R \subseteq \overline{M}_R$ such that $M_R \subseteq B_R \subseteq \overline{M}_R$. Let $f \in \mathrm{Hom}_R(B,\overline{M})$ such that $f(M) = 0$. Now E is injective and $B \subseteq \overline{M} \subseteq E$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & B_R \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[ddl, dashed, "f'"] \\
& \overline{M} \arrow[d, "i"'] & \\
& E &
\end{tikzcd}$$

The injectivity of E implies the existence of $f' \in \mathrm{Hom}_R(E,E)$ such that f' restricted to B_R is f. Since $f'(M) = f(M) = 0$, $f' \in M^{\wedge}$. Therefore, ker $f' \supseteq \overline{M} \supseteq B$ and $f'(B) = 0$. So $f(B) = 0$, and $\overline{M} \blacktriangledown M$.

(2) Show $\overline{M}$ contains each rational extension of M in E. Let $F_R \subseteq E_R$ such that $F \blacktriangledown M$. Let $t \in M^{\wedge}$ and $K = \{x \in F \mid t(x) \in M\}$. Assume $t(F) \neq 0$. Since $E \nabla M$, $t(F) \cap M \neq 0$. Therefore, there exists an $x^* \in F$ such that $0 \neq y = t(x^*) \in t(F) \cap M$. Now $0 \neq x^* \in K$ and $0 \neq t(x^*) \in t(K)$. Let $N = t(F) \cap M$. Then $K = t^{-1}(N) \cap F$, and is hence a submodule of F_R. Observe now that $M \subseteq K + M \subseteq F$. Therefore t_0, the restriction of t to $K + M$, is a function such that $t_0(K+M) \subseteq F$ because if $x \in K$, $t(x) \in M \subseteq F$. Since $F \blacktriangledown M$ and $t_0(M) = 0$, it follows that $t_0(K+M) = 0$. This implies $t(K) = t_0(K) = 0$. But, this is a contradiction to the fact that $0 \neq t(x^*) \in t(K)$. Hence, $t(F) = 0$ for all $t \in M^{\wedge}$. Therefore, by definition of $\overline{M}$, $F \subseteq \overline{M}$.

(3) Show $\overline{M}$ satisfies condition (b) of definition 1.2.1. Let $H \blacktriangledown M$. Let θ_K loosely denote the canonical injection of any module K into another set containing K. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\theta_M"] \arrow[d, "\theta_M"'] & H \arrow[ddl, dashed, "\phi"] \\ & \overline{M} \arrow[d, "\theta_{\overline{M}}"'] & \\ & E & \end{tikzcd}$$

Because E is injective, there exists $\phi \in \mathrm{Hom}(H,E)$ such that $\phi(M) = M$. But $H \blacktriangledown M$ implies $H \nabla M$. Therefore, $\ker \phi \cap M \subseteq \ker(\theta_{\overline{M}} \circ \theta_M) = 0$ since the injection maps are one-to-one. Because $\ker \phi \subseteq H$ and $H \nabla M$, this implies $\ker \phi = \{0\}$. Thus ϕ is one-to-one and H is R-isomorphic to $\phi(H)$. By 1.1.9, $\phi(H) \blacktriangledown \phi(M) = M$. Thus by part (2) of this proof, $\phi(H) \subseteq \overline{M}$. Therefore, ϕ is a monomorphism of H into $\overline{M}$ which extends the identity map on M.

(4) Show $\overline{M}$ satisfies part (c) of the definition of maximal rational extension. Suppose $H \supseteq \overline{M}$ and $H \blacktriangledown M$. Then $H \nabla M$ which implies $H \subseteq E$. Therefore, $H \subseteq \overline{M}$ and $H = \overline{M}$.

(5) If G_R is any maximal rational extension of M, we want to show the identity map of M into $\overline{M}$ can be extended to a monomorphism of G onto $\overline{M}$. Let G_R be any maximal rational extension of M. We know the injective hull of G, denoted by

$E(G)$, is an injective essential extension of M. Therefore, there exists an isomorphism $f : E(G) \to E$ such that f is the identity on M. Let M' be defined for $E(G)$ as $\overline{M}$ was for E. Since G is a rational extension of M and $G \subseteq E(G)$, $G \subseteq M'$ (parts(1), (2)). But by part 4, $M' = G$. Now $f(M') \subseteq \overline{M}$ since $f(M') \blacktriangledown f(M) = M$ by 1.1.9. Also, $f^{-1}(\overline{M}) \subseteq M'$ by the same reasoning. Therefore, $f(M') \subseteq \overline{M} = ff^{-1}(\overline{M}) \subseteq f(M')$ and $f(M') = \overline{M}$. Therefore, f induces as isomorphism from $M' = G$ onto $\overline{M}$ which is the identity on M.

<u>1.2.3 LEMMA</u>. Let R be a ring and E the injective hull of R_R. Let $\Lambda = Hom_R(E,E)$; then E is a left Λ-module.

Proof: Clearly Λ is a ring under composition. Define $\mu : \Lambda \times E \to E$ by $f \mu x = f(x)$. To show (E,μ) is a Λ-module, let f and g be elements of Λ and e_1 and e_2 be elements of E. One can easily check $f\mu(e_1+e_2) = (f\mu e_1) + (f\mu e_2)$ and $(f+g) \mu e_1 = (f\mu e_1) + (g\mu e_1)$. Also, $(f \circ g) \mu e_1 = (f \circ g)(e_1) = f(g(e_1)) = f\mu(g(e_1)) = f\mu(g\mu e_1)$ and hence, (E,μ) is a Λ-module.

<u>1.2.4 LEMMA</u>. Let $Q = Hom_\Lambda(E,E)$ where R is a ring and E is the injective hull of R_R; then E is a right Q-module. In fact, E is a (Λ,Q)-bimodule.

Proof: In this proof, we are assuming Q is a ring under composition that is opposite of the usual definition of composition of functions; that is, if $x \in E$ and $q,p \in Q$, then $(x)(q \circ p) = ((xq))p$. Let $x,y \in E$ and $p,q \in Q$. Define $\mu^* : E \times Q \to E$ by $x\mu^*q = xq$ where xq denotes x acted on by q. It is easy to show $(x+y)\mu^*q = x\mu^*q + y\mu^*q$ and $x\mu^*(q+p) = x\mu^*q + x\mu^*p$.

Also, $x\mu^*(q\circ p) = x(q\circ p) = ((xq))p = xq\mu^*p = (x\mu^*q)\mu^*p$. Therefore, (E,μ^*) is a Q-module. Let $f \varepsilon \Lambda$, $x \varepsilon E$, and $q \varepsilon Q$. Since q is a Λ-homomorphism, $(f\mu x)\mu^*q = (f\mu x)q = f(xq) = f\mu(x\mu^*q)$ and E is a (Λ,Q)-bimodule.

1.2.5 THEOREM. If R is a ring, E is the injective hull of R, and $\bar{R}$ is the maximal rational extension of R, then $\bar{R}$ is a ring whose operation $\cdot' : \bar{R} \times \bar{R} \to \bar{R}$ induces the module operation $\mu : \bar{R} \times R \to \bar{R}$ in $\bar{R}_R$; in particular, R is a subring of $\bar{R}$.

Proof: Let R be a ring and E the injective hull of R. Let $\bar{R}$ be defined as in 1.2.2 where $\Lambda = \mathrm{Hom}_R(E,E)$. Let $Q = \mathrm{Hom}_\Lambda(E,E)$. By 1.2.4, E is a (Λ,Q)-bimodule which implies for each $\lambda \varepsilon \Lambda$, $x \varepsilon E$, and $q \varepsilon Q$, $[\lambda(x)]q = \lambda[(x)q]$. If $R^\Lambda = \{\lambda \varepsilon \Lambda \mid \lambda(R) = 0\}$, then by 1.2.2, $\bar{R} = \{m \varepsilon E \mid R^\Lambda(m) = 0\} = \cap\{\ker \lambda \mid \lambda \varepsilon R^\Lambda\}$, and is the maximal rational extension of R_R. Observe since $1 \varepsilon R$, $R^\Lambda = \{\lambda \varepsilon \Lambda \mid \lambda(1) = 0\}$.

We first show Q is a right R-module. Define a function $\phi : R \to Q$ by $\phi(r) = \rho_r$ where $\rho_r(x) = xr$ for each $x \varepsilon E$ and module multiplication is understood. Clearly, $\rho_r \varepsilon Q$ for each $r \varepsilon R$. Let $r_1, r_2 \varepsilon R$ and $x \varepsilon E$. Then $(x)\rho_{r_1+r_2} = x(r_1+r_2) = xr_1 + xr_2 = (x)\rho_{r_1} + (x)\rho_{r_2}$ and hence, $\phi(r_1+r_2) = \phi(r_1) + \phi(r_2)$. Now $\phi(r_1\cdot r_2) = \rho_{r_1\cdot r_2}$ and $(x)\rho_{r_1\cdot r_2} = (xr_1)\cdot r_2 = (x)\rho_{r_1}\circ\rho_{r_2}$ which implies $\phi(r_1\cdot r_2) = \rho_{r_1\cdot r_2} = \rho_{r_1}\circ\rho_{r_2} = \phi(r_1)\circ\phi(r_2)$. Also, if $r_1 \neq 0$ and $\rho_{r_1} = 0$, then

$\rho_{r_1}(1) = 1 \cdot r_1 = 0$, which is bad. So, ϕ is a ring monomorphism, and Q contains a subring $R' = \phi(R)$ which is ring isomorphic to R. Define $\mu' : Q \times R \to Q$ by $q\mu' r = q \circ \phi(r) = q \circ \rho_r$ for each $q \in Q$ and $r \in R$. The ring Q, together with μ', is a R-module and moreover, Q is a ring whose multiplication (composition) induces the module multiplication, μ'.

Now we show Q_R is R-isomorphic to $\overline{R}_R$. If $q \in Q$, let $(1q) = q^\circ \in E$. Define $\theta : Q_R \to E_R$ by $\theta(q) = q^\circ$. If q_1 and q_2 are in Q, then $\theta(q_1+q_2) = 1(q_1+ q_2) = 1q_1 + 1q_2 = \theta(q_1) + \theta(q_2)$. If $r \in R$, $\theta(q\mu' r) = (qr)^\circ = 1(qr) = 1(q\circ\phi(r)) = 1(q\circ\rho_r) = (1q)\rho_r = (q^\circ)\rho_r = q^\circ \cdot r = \theta(q) \cdot r$. Thus θ is a R-homomorphism. We want to show θ is an R-isomorphism. Let $q \in Q$ such that $\theta(q) = q^\circ = 0$ and let $x \in E$; define, for all $r \in R$, a function $k : R \to E$ by $k(r) = xr$ where module multiplication is understood. Since E is injective, there is a $\lambda \in \Lambda$ such that λ restricted to R is k. Now $\lambda(1) = k(1) = x$. So, $xq = (\lambda(1))q = \lambda(1q) = \lambda(q^\circ) = \lambda(0) = 0$. Therefore, since x was arbitrarily chosen in E, $q = 0$; hence $\ker \theta = \{0\}$ and θ is a monomorphism. Next, we want to show $\theta(Q) = \overline{R}$. Let $q \in Q$ and $\lambda \in R^{\wedge}$. Then $q^\circ \in \theta(Q) = Q^\circ$ and $\lambda(q^\circ) = \lambda(1q) = (\lambda(1))q = (0)q = 0$. Since λ and q are arbitrary, $R^{\Lambda}(Q^\circ) = 0$ and $\theta(Q) = Q^\circ \subseteq \overline{R}$. Now we need to show $\overline{R} \subseteq Q^\circ$. Let $m \in E$. Define $\overline{\lambda}_m : R \to E$ by $\overline{\lambda}_m(r) = mr$ for each r in R. Since E is injective, there is a $\lambda_m \in \Lambda$ such that λ_m restricted to R

is $\overline{\lambda}_m$. Clearly, $\lambda_m(1) = m$ and, hence, for each m in E, there is a λ_m in Λ such that $\lambda_m(1) = m$. Fix $x \in \overline{R}$. Show $x \in Q^\circ$. Define $g : E \to E$ by $(m)g = \lambda_m(x)$ for each m in E. The function g is well-defined since if $\lambda'(1) = m$, then $(\lambda'-\lambda_m)(1) = 0$, which implies $\lambda' - \lambda_m \in R^\wedge$. Hence $(\lambda'-\lambda_m)(\overline{R}) = 0$ and then $(\lambda'-\lambda_m)(x) = 0$. So, $\lambda_m(x) = \lambda'(x)$ and g is well-defined. To show g is a Λ-homomorphism, let $\lambda' \in \Lambda$, $m \in E$ and show $(\lambda'(m))g = \lambda'(mg)$. By definition, $(\lambda'(m))g = \lambda_{\lambda'(m)}(x)$ and if $r \in R$, $\lambda_{\lambda'(m)}(r) = \lambda'(m) \cdot r = \lambda'(mr) = \lambda' \circ \lambda_m(r)$. So $(\lambda_{\lambda'(m)} - \lambda' \circ \lambda_m) \in R^\wedge$ which implies $\lambda_{\lambda'(m)}(y) = \lambda' \circ \lambda_m(y)$ for each $y \in \overline{R}$. Now $(\lambda'(m))g = \lambda_{\lambda'(m)}(x) = \lambda' \circ \lambda_m(x) = \lambda'((m)g)$, and $g \in Q$. Let i_E denote the identity on E; then $(\lambda_1 - i_E)(R) = 0$ and so $(\lambda_1 - i_E)(\overline{R}) = 0$. This means $\lambda_1(x) = i(x) = x$. So $x = \lambda_1(x) = 1g = g^\circ \in Q^\circ$ and $\overline{R} \subseteq Q^\circ$. Therefore, $\overline{R} = Q^\circ$ and Q is R-isomorphic to $\overline{R}$.

One can show $(\overline{R}, +, \cdot')$ is a ring where $(r_1 + r_2) = \theta(\theta^{-1}(r_1) + \theta^{-1}(r_2))$ and $r_1 \cdot' r_2 = \theta(\theta^{-1}(r_1) \circ \theta^{-1}(r_2))$ (where θ is the R-isomorphism between Q and $\overline{R}$). So, $\overline{R}$ is a ring whose ring multiplication induces the R-module multiplication, $\mu : \overline{R} \times R \to \overline{R}$, since if $r \in R$, then $\theta^{-1}(r) = \rho_r$ and $q^\circ \cdot' r = \theta(\theta^{-1}(q^\circ) \circ \theta^{-1}(r)) = \theta(q^\circ \rho_r) = \theta(q\mu' r) = \theta(q)\mu r = q^\circ \mu r$ for all $q^\circ \in \overline{R}$.

Let "." denote multiplication in a ring R. The right regular module, R_R, is constructed by defining $r\mu r' = r \cdot r'$. Note that the module multiplication of the injective hull of R_R

extends μ, and since $\overline{R}_R$ is a submodule of the injective hull, the module multiplication of $\overline{R}_R$ extends μ and hence multiplication in R. So, if $r_1, r_2 \in R$, $r_1 \cdot' r_2 = r_1 \mu r_2 = r_1 \cdot r_2$ and R is a subring of $\overline{R}$.

<u>1.2.6 THEOREM</u>. Let R be a ring and E the injective hull of R_R. If $Z(R) = 0$, then E is a ring whose multiplication operation $\cdot' : E \times E \to E$ induces the module multiplication $\mu : E \times R \to E$.

Proof: Let R be a ring with $Z(R) = 0$. Let $\overline{R}$ be defined as in 1.2.2 and 1.2.5. Since $\overline{R} \blacktriangledown R$ implies $\overline{R} \nabla R$, $\overline{R} \subseteq E$. However, $Z(R) = 0$ implies $E \blacktriangledown R$ which implies $E \subseteq \overline{R}$. Therefore, $\overline{R} = E$ and by 1.2.5, the theorem is proved.

The preceding theorems have proved that $Z(R) = 0$ is a sufficient condition for the injective hull to be a ring. The question "Is $Z(R) = 0$ a necessary condition" is answered in a paper by B. L. Osofsky [4]. In this paper, she gives an example of a ring with $Z(R) \neq 0$ whose injective hull is a ring. Thus, the answer to the above question is "no".

<u>1.2.7 THEOREM</u>. If R is a commutative ring, then $\overline{R}$ is a commutative ring.

Proof: Let $(R,+,\cdot)$ be a commutative ring and $(\overline{R},+,\cdot')$ be defined as in 1.2.5. By 1.2.6 we know R is a subring of $\overline{R}$. Fix $r \in R$. Define $\phi_1 : \overline{R} \to \overline{R}$ by $\phi_1(b) = r \cdot' b$ and $\phi_2 : \overline{R} \to \overline{R}$ by $\phi_2(b) = b \mu r$ for each $b \in \overline{R}$ where $\cdot'$ induces μ. To check that ϕ_1 and ϕ_2 are R-homomorphisms, let a and

$b \in \bar{R}$ and $r_1 \in R$. Then $\phi_1(a+b) = r \cdot'(a+b) = r \cdot' a + r \cdot' b =$ $\phi_1(a) + \phi_1(b)$ and $\phi_2(a+b) = (a+b)\mu r = a\mu r + b\mu r = \phi_2(a) + \phi_2(b)$. Also, $\phi_1(a\mu r_1) = r \cdot'(a\mu r_1) = r \cdot'(a \cdot' r_1) = (r \cdot' a) \cdot' r_1 =$ $\phi_1(a)\ \mu\ r_1$, and $\phi_2(a\mu r_1) = (a\mu r_1)\ \mu\ r = a\mu(r_1 \cdot r) = a\mu(r \cdot r_1) =$ $(a\mu r)\ \mu\ r_1 = \phi_2(a)\ \mu\ r_1$, since R is commutative. Therefore, ϕ_1 and $\phi_2 \in \mathrm{Hom}_R(\bar{R},\bar{R})$. Now let $z \in R$. Observe that $(\phi_1-\phi_2)(z) = \phi_1(z) - \phi_2(z) = r \cdot' z - z\ \mu\ r = r \cdot' z - z \cdot' r =$ $r \cdot z - z \cdot r = r \cdot z - r \cdot z = 0$, since R is a subring of $\bar{R}$ and R is commutative. Therefore, if $z \in R$, $(\phi_1-\phi_2)(z) = 0$ and $(\phi_1-\phi_2)(R) = 0$. However, $\bar{R} \blacktriangledown R$ implies $(\phi_1-\phi_2)(b) = 0$ for each $b \in \bar{R}$ which implies $r \cdot' b = b\ \mu\ r = b \cdot' r$ for each $b\ \mu\ \bar{R}$. Since r was arbitrarily chosen, we know $r \cdot' b =$ $b \cdot' r$ for each $r \in R$, $b \in \bar{R}$; that is, elements of R commute with elements of $\bar{R}$.

Now we are ready to show $\bar{R}$ is commutative. Fix $\bar{r} \in \bar{R}$. Define $f : \bar{R} \to \bar{R}$ by $f(b) = \bar{r} \cdot' b$ for each b in $\bar{R}$ and $g : \bar{R} \to \bar{R}$ by $g(b) = b \cdot' \bar{r}$ for each $b \in \bar{R}$. We must show $f,g \in \mathrm{Hom}_R(\bar{R},\bar{R})$. The additive property clearly works. If $r \in R$, $f(b\mu r) = \bar{r} \cdot'(b\mu r) = \bar{r} \cdot'(b \cdot' r) = (\bar{r} \cdot' b) \cdot' r = f(b)\mu r$ and $g(b\mu r) = (b\mu r) \cdot' \bar{r} = (b \cdot' r) \cdot' \bar{r} = b \cdot'(r \cdot' \bar{r}) = b \cdot'(\bar{r} \cdot' r) =$ $(b \cdot' \bar{r}) \cdot' r = g(b)\ \mu\ r$, since R commutes with $\bar{R}$. Thus, f and g are R-homomorphisms. Let $r \in R$. We see that $(f-g)(r) =$ $\bar{r} \cdot' r - r \cdot' \bar{r} = \bar{r} \cdot' r - \bar{r} \cdot' r = 0$. Therefore, $(f-g)(R) = 0$, which implies (since $\bar{R} \blacktriangledown R$) that $(f-g)(\bar{R}) = 0$; that is, $\bar{r} \cdot b =$

$b \cdot \bar{r}$ for all $b \in \bar{R}$. Since $\bar{r}$ was arbitrarily chosen, this is true for each $\bar{r} \in \bar{R}$. This means $\bar{R}$ is commutative.

Section 3: Utumi Quotient Rings and von Neumann Rings

If R is a ring such that $Z(R) = 0$, in this section we will show that the injective hull of R_R is a maximal right quotient ring of R. We will also show that under this condition, the injective hull of R_R is a right self-injective, von Neumann ring.

1.3.1 DEFINITION. A ring A containing a ring R is a (Utumi) right quotient ring of R in case A_R is a rational extension of R_R.

1.3.2 DEFINITION. A right quotient ring A of R is maximal in case given any right quotient ring T of R, there exists a ring monomorphism of T into A which induces the identity map on R.

In the proof of 1.2.5, R is shown to be a subring of $\bar{R}$ and we already know $\bar{R}$ is a rational extension of R; hence, $\bar{R}$ is a right quotient ring of R. Therefore, every ring has a right quotient ring.

1.3.3 LEMMA. If A and B are right quotient rings of R and if $\phi : A_R \to B_R$ is a module monomorphism which induces the identity map on R, then ϕ is a ring monomorphism of A into B.

Proof: Let A and B be right quotient rings of R and ϕ be a module monomorphism of A_R into B_R which induces the identity map on R. Let "$\circ$" denote the product in A and "$\cdot$" denote the product in B. If $a \in A$ and $r \in R$, then $\phi(a \circ r) =$

$\phi(a) \cdot r$. Since ϕ is a module monomorphism, consider A to actually be a submodule of B; that is, identify a and $\phi(a)$. Then $a \circ r = a \cdot r$ for all $a \in A$, $r \in R$. Let $a \in A$. Define $f_1 : A_R \to A_R$ by $f_1(x) = a \circ x$ for all $x \in A$ and $f_2 : A_R \to B_R$ by $f_2(x) = a \cdot x$ for all $x \in A$. Clearly, f_1 and f_2 are well-defined R-homomorphisms and so $f = f_1 - f_2$ is. Let $r \in R$. Then $f(r) = (f_1 - f_2)(r) = a \circ r - a \cdot r = 0$ and hence $f(R) = 0$. However, $B \blacktriangledown R$ then implies $f(A) = 0$ which implies $f(a_1) = a \circ a_1 - a \cdot a_1 = 0$ for all a_1 in A. Therefore, $a \circ a_1 = a \cdot a_1$ for all a_1 in A and since a was chosen arbitrarily, $a \circ a_1 = a \cdot a_1$ for all a and a_1 in A. Hence, $\phi(a \circ b) = a \circ b = a \cdot b = \phi(a) \cdot \phi(b)$ for all a and b in A, which implies ϕ is a ring monomorphism.

<u>1.3.4 THEOREM</u>. If A is a right quotient ring of R, there exists a ring monomorphism ϕ of A into $\bar{R}$ which induces the identity map on R.

Proof: Let A be a right quotient ring of R. Then since $A \blacktriangledown R$, $A \nabla R$ and therefore $A \subseteq E(R)$, the injective hull of R. By 1.2.2, $\bar{R}$ is a maximal rational extension of R in $E(R)$. Thus $i : R \to R$ can be extended to a module monomorphism $\phi : A_R \to \bar{R}_R$. But, by 1.3.3, ϕ is a ring monomorphism.

The preceding theorem tells us if R is a ring, $\bar{R}$ is a maximal right quotient ring of R. So if $Z(R) = 0$, $E(R)$ is a maximal right quotient ring of R by 1.2.6.

1.3.5 COROLLARY. If R is a ring, then any two maximal right quotient rings of R are ring isomorphic under a mapping that induces the identity mapping on R.

Proof: Let A and B be maximal right quotient rings of a ring R. Then by definition of maximal right quotient ring, there exist ring monomorphisms ϕ_1 and ϕ_2 such that $\phi_1 : A \to B$ and $\phi_2 : B \to A$ which induce the identity map on R. Let i_A and i_B denote the identity maps on A and B, respectively. Now $(\phi_2 \circ \phi_1 - i_A)(R) = 0$, which implies, since $A \blacktriangledown R$, that $(\phi_2 \circ \phi_1 - i_A)(A) = 0$ which implies $\phi_2 \circ \phi_1 = i_A$. Likewise, $\phi_1 \circ \phi_2 = i_B$. Hence ϕ_1 is a ring isomorphism.

1.3.6 DEFINITION. A ring R is said to be von Neumann provided for each $a \in R$, there exists $x \in R$ such that $axa = a$.

1.3.7 DEFINITION. Let R be a ring. The Jacobson Radical of R, denote by $J(R)$, is the intersection of all the maximal right ideals of R.

1.3.8 DEFINITION. An element x of a ring R is said to be right quasi-regular provided there exists an element $x' \in R$ such that $x + x' + xx' = 0$ or equivalently, $(1+x)(1+x') = 1$.

The definitions of left quasi-regular and quasi-regular are analogus. A two-sided ideal I of R is said to be left quasi-regular provided every element of I is left quasi-regular. It is well-known that if R is a ring, $J(R)$ is a left (and right) quasi-regular two-sided ideal containing each left quasi-regular two-sided ideal of R. The proofs will be omitted here but may be found in [3].

1.3.9 DEFINITION. A module M_R is said to be quasi-injective in case each homomorphism of any submodule of M_R into M_R can be extended to a homomorphism of M_R into M_R.

The following lemmas are proved in order to show the endomorphism ring, Λ, of a quasi-injective module is a right self-injective, von Neumann ring. We will use this theorem to prove that under certain conditions the injective hull of a ring is von Neumann and right self-injective.

1.3.10 LEMMA. A ring R is von Neumann if and only if for each $a \in R$, there exists an $e^2 = e \in R$ such that $aR = eR$.

Proof: ($\rightarrow$) Let R be a von Neumann ring. Choose $a \in R$. Then there exists $x \in R$ such that $axa = a$. Then $e = ax$ satisfies $e^2 = e$ and $eR = axR \subseteq aR$. But $a = ea \in eR$ so that $aR \subseteq eR$ and hence, $aR = eR$, where e is an idempotent.

($\leftarrow$) Suppose $aR = eR$ for all $a \in R$, where e is an idempotent and depends on a. Let $a \in R$. Then there is $e^2 = e \in R$ such that $aR = eR$. So, there exists $r \in R$ such that $a = er$ and there exists $r' \in R$ such that $e = ar'$. Let $x = r'$. Then $axa = (a)(r')(er) = (ar')(er) = (e)(er) = e^2r = er = a$. Hence, R is von Neumann.

1.3.11 LEMMA. Let S be a von Neumann ring and let $P_r(S)$ denote the set of principal right ideals of S. Then

(1) if A and B are elements of $P_r(S)$, $A + B \in P_r(S)$;

(2) $P_r(S)$ contains each finitely generated right ideal of S.

Proof: (1) Let A and B be in $P_r(S)$. By 1.3.10, there exists $e = e^2$ in S and $f = f^2$ in S such that $A = eS$ and $B = fS$. Let $B_1 = (1-e)fS$. Now $A + B = \{eu + fv \mid u,v \in S\}$ and $A + B_1 = \{e(u'-fv) + fv \mid u',v \in S\}$. Choose $x \in A + B$. Then $x = eu + fv = eu + efv - efv + fv = e(u+fv-fv)+fv$, which implies $x \in A + B_1$. If $x \in A + B_1$, then $x = eu' + fv - efv = e(u'-fv) + fv$, which implies $x \in A + B$. Now, $B_1 = f_1S$ where $f_1 = (f-ef) = (1-e)f$, and ${f_1}^2 = f_1 \in S$ and since $f_1 \in B_1$, $ef_1 = e(1-e)f = (e-e^2)f = 0$. Now let $f' = f_1(1-e)$. Then $f' \cdot f_1 = f_1(1-e)f_1 = f_1f_1 - f_1ef_1 = {f_1}^2 = f_1$ and hence $f' \cdot f_1 = f_1$. So $(f')^2 = f'(f_1(1-e)) = f_1(1-e) = f'$. Since $f' = f_1(1-e) \in f_1S$ and $f_1 = f'f_1 \in f'S$, it follows that $f'(S) = f_1(S) = B_1$. Hence, $A + B = eS + f'S$. Because $ef' = 0$ and $f'e = 0$, $A + B = eS + f'S = (e+f')(S) \in P_r(S)$.

(2) Suppose K is a finitely generated right ideal of S. Then there exist $a_1, a_2, \ldots, a_n$ in S such that $K = a_1S + \ldots + a_nS$. But, 1.3.10 implies $a_iS = e_iS$, where e_i is an idempotent of S. Part (1) of this proof can be extended to finite sums. Hence K is a principal right ideal and is generated by an idempotent.

<u>1.3.12 THEOREM</u>. Let M_R be a quasi-injective module, $\Lambda = \mathrm{Hom}_R(M,M)$, and $J = J(\Lambda)$. Then

(1) $J = \{\lambda \in \Lambda \mid M \nabla \ker \lambda\}$ and Λ/J is von Neumann;

(2) if $J(\Lambda) = 0$, then Λ is a right self-injective, von Neumann ring.

Proof: Let $I = \{\lambda \in \Lambda \mid M \nabla \ker \lambda\}$. Let $\lambda \in \Lambda$ and $\alpha, \mu \in I$. One can easily show $\ker(\mu - \alpha) \supseteq \ker \mu \cap \ker \alpha$. Since $M \nabla (\ker \mu \cap \ker \alpha)$, $M \nabla \ker(\mu - \alpha)$, and thus $\mu - \alpha \in I$. Also, $\ker(\lambda \circ \alpha) \supseteq \ker \alpha$, which implies $M \nabla (\ker \lambda \circ \alpha)$. Hence, $\lambda \circ \alpha \in I$ and I is a left ideal of Λ. Let $0 \neq x \in M$. We wish to show $xR \cap \ker(\alpha \circ \lambda) \neq 0$. If $\lambda(xR) \neq 0$, then, because $\lambda(xR)$ is a submodule of M, $\lambda(xR) \cap \ker \alpha \neq 0$ since $M \nabla \ker \alpha$. Therefore, there exists $0 \neq xr \in xR$ such that $\lambda(xr) \neq 0$ and $\alpha(\lambda(xr)) = 0$. However, this means $0 \neq xr$ is in $\ker(\alpha \circ \lambda) \cap xR$. If $\lambda(xR) = 0$, then $\alpha(\lambda(xR)) = 0$. This implies $0 \neq xR \subseteq \ker(\alpha \circ \lambda)$ and so $xR \cap \ker(\alpha \circ \lambda) \neq 0$. In either case, $M \nabla \ker(\alpha \circ \lambda)$. Thus $\alpha \circ \lambda \in I$ and we can conclude that I is a two-sided ideal of Λ.

Let $\lambda \in I$. If $x \in \ker \lambda \cap \ker(1+\lambda)$, then $\lambda(x) = 0$ and $x + \lambda(x) = 0$, which implies $x = 0$. Since $M \nabla \ker \lambda$, $\ker(1+\lambda) = 0$. Thus we know for each $\lambda \in I$, $1 + \lambda$ is a one-to-one function and has a left inverse. Therefore, I is a left quasi-regular, two-sided ideal of R and consequently is a subset of J.

Now let $\lambda \in \Lambda$ and L_R be a submodule of M_R, maximal with respect to $N \cap \ker \lambda = 0$, where N is a submodule of M_R; let $K = \ker \lambda$. Consider the relation ϕ, mapping $\lambda(L)$ into L, defined by $\phi(\lambda(x)) = x$ for each $\lambda(x) \in \lambda(L)$. If $x, y \in L$

such that $\lambda(x) = \lambda(y)$, then $\lambda(x-y) = 0$ and $x-y \in L \cap K = 0$. So $x = y$ and ϕ is a function. Since M is quasi-injective and $\lambda(L)_R \subseteq M_R$, ϕ is induced by some $\theta \in \Lambda$. Let $x + y = u \in L + K$ where $x \in L$ and $y \in K$. It is clear that $L + K \subseteq \ker(\lambda-\lambda\theta\lambda)$ since $(\lambda-\lambda\theta\lambda)(u) = \lambda(u) - \lambda\theta\lambda(u) = \lambda(x) - \lambda(x) = 0$.

Claim: $M \nabla (L+K)$.

Subproof: Assume $L + K$ is not essential in M. There exists a $N_R \subseteq M_R$ such that $(L+K) \cap N = 0$. We claim now that $(L+N) \cap K = 0$. Let $x \in L$ and $y \in N$ such that $0 \neq x+y \in (L+N) \cap K$. Then $(x+y) - x = y \in N \cap (L+K) = 0$. Therefore, $0 \neq x+y = x \in K \cap L = 0$, which is a contradiction since L is maximal with respect to missing K. Hence, $M \nabla (L+K)$.

Therefore $\lambda-\lambda\theta\lambda \in I$, which means for all $\lambda + I \in \Lambda/I$, there exists $\theta \in \Lambda$ such that $\lambda + I = \lambda\theta\lambda + I$. Hence, Λ/I is a von Neumann ring.

To finish part (1), it suffices to show $J \subseteq I$. Let $\lambda \in J$. By the previous argument, there exists a $\theta \in \Lambda$ such that $\lambda-\lambda\theta\lambda \in I$. Since J is left quasi-regular and $-\lambda\theta \in J$, there exists a $y \in \Lambda$ such that $y \circ (1+-\lambda\theta) = 1$. But $\lambda = y \circ (1-\lambda\theta)\circ\lambda = y \circ(\lambda-\lambda\theta\lambda) \in I$, since I is an ideal. Therefore, $J \subseteq I$ and thus $J = I$ and (1) is proved.

(2) It follows from part (1) if $J = 0$, then $\Lambda/J = \Lambda/0 = \Lambda$ is a von Neumann ring. Lemma 1.3.11 will be used in the remainder of the proof to show Λ is right self-injective.

Let f be a Λ-homomorphism from a right ideal I of Λ into Λ. Define IM to be the submodule of M_R generated by $\{\lambda(m) \mid \lambda \in I$ and $m \in M\}$. If $x \in IM$, then there exist $n \in Z^+, \lambda_1 \cdot \cdot \cdot \lambda_n \in I$ and $m_1 \cdot \cdot \cdot m_n \in M$ such that $x = \sum_{i=1}^{n} \lambda_i(m_i)$. Consider the function θ between IM and M defined by $\theta(x) = \sum_{i=1}^{n} f(\lambda_i)(m_i)$. We need to show θ is indeed a function. Suppose $y \in IM$. Then $y = \sum_{j=1}^{t} \mu_j(m_j)$, where $\mu_j \in I$ and $m_j \in M$ for $1 \leq j \leq t$. The right ideal of Λ generated by $\{\lambda_1, \lambda_2, \cdot \cdot \cdot \lambda_n, \mu_1, u_2, \cdot \cdot \cdot \mu_t\}$ is of the form $e\Lambda$ where $e^2 = e \in \Lambda$. Observe since $\lambda_1 \in e\Lambda$, $\lambda_1 = e\lambda$ for some $\lambda \in \Lambda$ and $e\lambda_1 \in e\Lambda$. Also, $e\lambda_1 = e(e\lambda) = e\lambda = \lambda_1$. Likewise $e\mu_j = \mu_j$, $f(\lambda_i) = f(e)\lambda_i$, and $f(\mu_j) = f(e)\cdot\mu_j$. Consequently, $\theta(x) = \sum_{i=1}^{n} f(\lambda_i) m_i = \sum_{i=1}^{n} (f(e)\lambda_i)(m_i) = f(e) \sum_{i=1}^{n} \lambda_i(m_i) = f(e)\cdot x$. Similiarly, $\theta(y) = \sum_{j=1}^{t} f(\mu_j)(m_j) = f(e)\cdot y$. Thus if $x = y$, then $\theta(x) = f(e)\cdot x = f(e)\cdot y = \theta(y)$ and θ is a function. Clearly, θ is an R-homomorphism from IM into M. Since M is quasi-injective, θ is induced by some element $\theta' \in \Lambda$. Then $(\theta' \circ \lambda)(m) = \theta'(\lambda(m)) = f(\lambda)(m)$ for all $m \in M$ and $\lambda \in I$, so that $f(\lambda) = \theta' \circ \lambda$ for all $\lambda \in I$. This establishes that Λ satisfies Baer's condition for injectivity, and we conclude Λ is self-injective as a right Λ-module.

<u>1.3.13 LEMMA</u>. If R is a ring such that $Z(R) = 0$ and if E is the injective hull of R, then $\Lambda = \mathrm{Hom}_R(E,E)$ is a right self-injective, von Neumann ring.

Proof: Let R be a ring such that $Z(R) = 0$. Let E denote the injective hull of R_R and $\Lambda = \mathrm{Hom}_R(E,E)$. Let $\lambda \in J(\Lambda)$.

By 1.3.12, since injective implies quasi-injective, $E \nabla \ker \lambda$. But $\lambda(\ker \lambda) = 0$ which implies, since $E \blacktriangledown \ker \lambda$, that $\lambda(E) = 0$ or that $\lambda = 0$. Hence, $J(V) = 0$ and by 1.3.12, Λ is a right self-injective, von Neumann ring.

1.3.14 THEOREM. If R is a ring such that $Z(R) = 0$ and if E is the injective hull of R, then $\Lambda = \mathrm{Hom}_R(E,E)$ is ring isomorphic to E.

Proof: Let R be a ring such that $Z(R) = 0$. Let E denote the injective hull of R_R and $\Lambda = \mathrm{Hom}_R(E,E)$. We first want to show Λ is a right R-module. Let $r \in R$. Define $\eta_r : R_R \to E_R$ by $\eta_r(x) = r \cdot x$ for all $x \in R$, where "$\cdot$" is multiplication in E. Since E_R is injective and η_r is a R-homomorphism, η_r is induced by some $\eta_r{}^* \in \Lambda$. Define $\phi : R \in \Lambda$ by $\phi(r) = \eta_r{}^*$. Let $a,b \in R$. One can show without much difficulty that $[\eta^*_{a+b} - (\eta^*_a + \eta^*_b)](R) = 0$ and $(\eta^*_{a\cdot b} - \eta^*_a \circ \eta^*_b)(R) = 0$. Because $Z(R) = 0$, $E \blacktriangledown R$ and therefore $[\eta^*_{a+b} - (\eta^*_a + \eta^*_b)](E) = 0$ and $[\eta^*_{a\cdot b} - \eta^*_a \circ \eta^*_b](E) = 0$. Hence $\eta^*_{a+b} = \eta^*_a + \eta^*_b$ and $\eta^*_{a\cdot b} = \eta^*_a \circ \eta^*_b$ and ϕ is a ring homomorphism. Assume $r \in \ker \phi$; then $\eta^*_r(x) = 0$ for all $x \in E$. However, $1 \in R \subseteq E$; so, $\eta^*_r(1) = r = r \cdot 1 = 0$. Thus we know $\ker \phi = \{0\}$. Hence, Λ contains a subring isomorphic to R, namely $\phi(R)$. The ring Λ is a natural right R-module under the operation μ, where $\lambda \mu r = \lambda \circ \eta^*_r$ for all $\lambda \in \Lambda$ and $r \in R$. As in the proof of 1.2.5, for each $x \in E$, there is a $\eta^*_x \in \Lambda$ such that $\eta^*_x(1) = x$. Define a function $\theta : E \to \Lambda$ by $\theta(x) = \eta^*_x$. Clearly

θ is well-defined and $\theta(x+y) = \eta^*_{x+y} = \eta^*_x + \eta^*_y$ since $[\eta^*_{x+y} - (\eta^*_x+\eta^*_y)](r) = (x+y)r - xr - yr = 0$ for every $r \in R$ and $E \blacktriangledown R$. It is not hard to see that $\eta^*_{x\cdot r} - \eta^*_x\mu r \in \mathrm{Hom}_R(E,E)$ and that $(\eta^*_{x\cdot r} - \eta^*_x\mu r)(1) = 0$. Thus, $(\eta^*_{x\cdot r} - \eta^*_x\mu r)(R) = 0$, which implies $(\eta^*_{x\cdot r} - \eta^*_x\mu r)(E) = 0$. So, $\theta(x\cdot r) = \eta^*_{x\cdot r} = \eta_x\mu r = \theta(x)\mu r$ and θ is an R-homomorphism. If $\theta(x) = \theta(y)$, then $x = \eta^*_x(1) = \eta^*_y(1) = y$ and θ is one-to-one. Choose $\lambda \in \Lambda$. Let $y = \lambda(1)$. Then $(\eta^*_y - \lambda) \in \mathrm{Hom}_R(E,E)$, and $(\eta^*_y - \lambda)(1) = 0$, which implies $(\eta_y - \lambda)(R) = 0$. But again, $E \blacktriangledown R$ implies $(\eta^*_y - \lambda)(E) = 0$, which means $\theta(y) = \eta_y = \lambda$. Hence θ is a one-to-one, onto module homomorphism between E and Λ. So, Λ is a rational extension of R, where we identify R with $\phi(R)$, and Λ is thus a right quotient ring of R. Since θ is the identity on R, by 1.3.3 θ is a ring isomorphism and we can conclude E and Λ are ring isomorphic.

<u>1.3.15 THEOREM</u>. Let R be a ring such that $Z(R) = 0$. Then the injective hull of R_R is a right self-injective, von Neumann ring which is also the maximal right quotient ring of R.

Proof: Let R be a ring such that $Z(R) = 0$. Let E denote the injective hull of R_R. By 1.2.6, $E = \bar{R}$ is a ring that contains R as a subring and is the maximal rational extension of R_R, hence, by 1.3.4 the maximal right quotient ring of R. By 1.3.13, Λ is a right self-injective, von Neumann ring and so, by 1.3.14, E is also.

CHAPTER II

THE INJECTIVE ENVELOPE OF THE $n \times n$ UPPER TRIANGULAR MATRIX RING

This chapter is devoted to showing the injective hull of the $n \times n$ upper triangular matrix ring is the full $n \times n$ matrix ring.

In the following proofs, let D represent a division ring, $M_n(D) = S$ denote the full $n \times n$ matrix ring over D, and $UT_n(D) = K$ be the $n \times n$ upper triangular matrix ring over D. The matrix with 1 in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column with zero everywhere else will be denoted by E_{ij}. Also, $E(K)$ will denote the injective hull of K.

<u>2.1 LEMMA</u>. $Z(K) = 0$.

Proof: Define $S_{in} = \{a \cdot E_{in} \mid a \in D\}$; we want to show that S_{in} is a simple submodule of K_K for $1 \le i \le n$. Clearly, S_{in} is closed under addition. Let $k \in K$ and α_{nn} be that element of D that belongs to the $n^{\underline{th}}$ row and $n^{\underline{th}}$ column of k. Let $s \in S_{in}$. Then $s = a \cdot E_{in}$ for some $a \in D$. But $s \cdot k$ is the matrix with $a \cdot \alpha_{nn}$ in the $i^{\underline{th}}$ row and $n^{\underline{th}}$ column with zero everywhere else. So $s \cdot k \in S_{in}$ and S_{in} is a submodule of K_K. Let $0 \ne L_K$ be a submodule of S_{in}; we want to show $L = S_{in}$. Because $(S_{in})_K \subseteq K_K$, $L_K \subseteq K_K$. Choose $0 \ne b \in L$. Then $b = a \cdot E_{in}$

for some $a \in D$. Let $b' = a^{-1} \cdot E_{nn}$. Then $b' \in K$ and $E_{in} = b \cdot b' \in L$. Thus, $S_{in} = (b \cdot b')S_{in} \subseteq L$ which implies $S_{in} = L$. Therefore, S_{in} is simple for each $1 \leq i \leq n$.

It is clear that if I is an essential right ideal of K, then $I \cap S_{in} = S_{in}$ because S_{in} is simple for each $1 \leq i \leq n$. This implies $S_{in} \subseteq I$ for each $1 \leq i \leq n$ and hence, $\sum_{i=1}^{n} S_{in} \subseteq I$.

Observe if c is any element of K, then $c \cdot E_{in}$ is the matrix whose $n^{\underline{th}}$ column is equal to the $i^{\underline{th}}$ column of c and is zero everywhere else.

Assume $a \in Z(K)$. Thus $\sum_{i=1}^{n} S_{in} \subseteq (0:a)$ as shown above. This implies $a \cdot \sum_{i=1}^{n} S_{in} = 0$ and therefore, $a \cdot E_{in} = 0$ for each $1 \leq i \leq n$. Thus, the matrix a must be equal to zero. Therefore, $Z(K) = 0$.

2.2 LEMMA. $S_K \triangledown K_K$.

Proof: It is not hard to show S is a K-module since K is a subring of S. To show $S_K \triangledown K_K$, let $0 \neq x \in S$ and prove $K \cap xK \neq 0$. Since $x \neq 0$, there exists an integer j, $1 \leq j \leq n$, such that the $j^{\underline{th}}$ column of x is not zero. So, $x \cdot E_{jn}$ is the matrix with $n^{\underline{th}}$ column equal to the $j^{\underline{th}}$ column of x and zero everywhere else. Thus $x \cdot E_{jn} \neq 0$ and is an element of K. Now, $E_{jn} \in K$ and $x \cdot E_{jn} \in xK$. So $0 \neq x \cdot E_{jn} \in xK \cap K$ and $S_K \triangledown K_K$.

2.3 THEOREM. $E(K)$ is a right S-module.

Proof: Since K is a ring and $Z(K) = 0$, by 1.2.6 $E(K)$ is a ring. Since by 2.2, $S_K \nabla K_K$, there exists a function $\phi \in \mathrm{Hom}_K(S,E(K))$ such that $\phi(k) = k$ for each $k \in K$. We will show $E(K)$ is a S-module by first showing multiplication in S is equivalent to multiplication in $E(K)$. Consider the following diagram where θ is the canonical injection map:

$$\begin{tikzcd} 0 \arrow[r] & K_K \arrow[r, "\theta_K"] \arrow[d, "\theta_K"'] & S_K \arrow[dl, dashed, "\phi"] \\ & E(K)_K & \end{tikzcd}$$

We need to show $\phi(ss') = \phi(s)\phi(s')$ for all $s,s' \in S$. Fix $s \in S$. Define $\psi \in \mathrm{Hom}_K(S,E(K))$ by $\psi = \phi \circ \lambda_s$ where $\lambda_s(s') = ss'$ for each $s' \in S$. Define $\psi' \in \mathrm{Hom}_K(S,E(K))$ by $\psi' = \lambda_{\phi(s)} \circ \phi$ where $\lambda_{\phi(s)}(e) = \phi(s)\cdot e$ for all $e \in E(K)$. However, if $k \in K$, $(\psi-\psi')(k) = \psi(k) - \psi'(k) = (\phi\circ\lambda_s)(k) - (\lambda_{\phi(s)}\circ\phi)(k) = \phi(s\cdot k) - \phi(s)\cdot k = \phi(s)\cdot k - \phi(s)\cdot k = 0$. Therefore, $(\psi-\psi')(K) = 0$ which implies, since $(\psi-\psi') \in \mathrm{Hom}_K(S,E(K))$ and $E(K) \blacktriangledown K$, that $(\psi-\psi')(S) = 0$. Thus, $\psi(s) = \psi'(s)$ or equivalently if $s \in S$, $\phi(ss') = \phi(s)\phi(s')$. Since s was arbitrary, this is true for each $s \in S$, and ϕ is a ring homomorphism between S and $E(K)$. So, $E(K)$ can be shown to be an S-module by defining $e\mu s = e \bullet \phi(s)$ for all $e \in E(K)$, $s \in S$.

If one considers S_K to actually be a submodule of $E(K)_K$, then to prove $E(K)$ is an S-module one must show $s\cdot s' = s\cdot' s'$

where $\cdot$ is multiplication in S and $\cdot'$ is multiplication in $E(K)$.

<u>2.4 THEOREM</u>. $E(K) = S$.

Proof: By the Wedderburn Theorem, $S = M_n(D)$ is a self-injective right S-module. From the preceding lemmas, we know $E(K)_S \supseteq S_S$. Thus, $E(K)_S = S_S \oplus S'_S$. But any S-module is a K-module since K is a subring of S. Thus, $E(K)_K = S_K \oplus S'_K$. However, since $K \subseteq S$, $K \cap S' = 0$ which implies $S' = 0$, since $E(K) \nabla K$. Therefore, $E(K) = S$.

SUMMARY

In conclusion, we have shown if the right singular ideal of a ring R is zero, then the injective hull of R is a ring whose operations preserve the module structure of the injective hull. Furthermore, this ring is right self-injective and von Neumann.

These results are used to show the injective hull of the $n \times n$ upper triangular matrix ring over a division ring is the full $n \times n$ matrix ring. For an alternate proof of this (nontrivial) result, the reader may consult [1].

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